Smooth Structures on Spheres

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Working from [KM63], [Kos93, VIII. 5.6], [Lan00], [Mil], [LL12].

1 Background

1.1 Notions of Equivalence

A smooth structure is a choice of atlas, however *two inequivalent atlases can be diffeomorphic.* That is here we are interested in uniqueness of the smooth structure up to diffeomorphism, not equivalence of atlases, and these two things are *different.* For two atlases to be equivalent we need the identity to be a diffeomorphism, however for two spaces to be diffeomorphic we just need some map to be a diffeomorphism.

The literal first exercise I did in differential geometry is check that the following two charts on $\mathbb{R}$ are not equivlaent

$$x \mapsto x, \quad x \mapsto x^{\frac{1}{3}}$$

however these two charts (or atlases) do define diffeomorphic manifold structures on $\mathbb{R}$, the diffeomorphis is simply $x \mapsto x^3$.

1.2 Diffeomorphism with Boundary

First we note that D^n is a manifold with boundary. Moreover it is oriented. Thus we need to be careful about what we mean by diffeomorphism. First recall the definition of smooth maps between manifolds with boundary. A map between open sets of upper half space

$$U \subseteq \mathbb{H}^n \to V \subseteq \mathbb{H}^m$$

is smooth if it admits a smooth extension in the domain and is smooth when the image is considered as inside $\mathbb{R}^m$.

$$f : M \to N$$

is smooth at a point $p \in M$ if there are charts around p and $f(p)$, φ, ψ respectively, such that the composition $\psi \circ f \circ \varphi^{-1}$ is smooth between the open subsets of $\mathbb{H}^n$ and $\mathbb{H}^m$.

1.3 Smooth Homotopies

Smooth maps can be identified up to

- Psuedo-isotopy: $f_0, f_1 : M \to M$ are psuedo-isotopic if there is an $F :: I \times M \to I \times M$ that is a diffeomorphism such that $F(0, -) = f_0$ and $F(1, 0) = f_1$.
- Isotopy: A psuedo-sotopy is an isotopy if in addition $F(t, M) = \{t\} \times M$, that is it is "level preserving".

We denote π_0Diff(M) the set of maps up to isotopy, and $\tilde{\pi}_0$Diff(M) the set of maps up to psuedo-isotopy. There is also smooth homotopy, which is weaker, it is a smooth map that is a homotopy. You can also mix and match other conditions. In particular an isotopy is a homotopy.

Lemma (Lee somewhere)**.** *Gluing smooth manifolds along a diffeomorphism returns a smooth manifold.*

Lemma. *Simply connected implies orientable.*

It follows from some covering space argument.

Lemma ([LL12], Prop 15.9)**.** *A connected orientable manifold has exactly two orientations.*

2 h-Cobordism and its Consequences

Smale proved a version of the h-cobordism theorem (for even dimensions I think) in [Sma], to prove the generalized Poincare conjecture (that homotopy spheres are homeomorphic to spheres). In [Sma62] he gave a much more uniform treatment, proving the following theorem:

Theorem (Thm 4.1 loc. cit.)**.** *Consider a submanifold $M^k \subseteq W^n$ where W is connected and compact, M is closed and both are simply connected, $n > 5$. If M is a deformation retract of W then W is diffeomorphic to a tubular neigbhourhood of M in W.*

This is the most conceptual and also most powerful h-cobordism variant, and has bountiful corollaries.

Corollary (h-Cobordism Theorem)**.** *Let W be a simply connected $n > 5$ h-cobordism between M and N (both compact), then it is diffeomorphic to $M \times [0, 1]$.*

Proof. Apply 4.1 to one of the boundary components. Because the boundary is co-dimension 1 a tubular neighborhood is given by crossing with $D^1 = [0, 1]$.

Corollary. *For closed simply connected $n > 5$ manifoldsM and N are h-cobordant if and only if they are diffeomorphic.*

Proof. If two such spaces are diffeomorphic, say $f : M \to N$, they are h-cobordant by taking $M \times [0, 1]$ as a cobordism $M \simeq M$ and $N \times [0, 1]$ similarly, then gluing these cylinders along f gives us the required cobordism $M \simeq N$.

If $M \simeq N$ are cobordant then by the process described in [KM63] we can perform spherical modifications to produce a simply connected cobordism which is by definition compact, call it W. Hence the h-cobordism theorems conditions are fulfilled and we see that $W \cong M \times [0, 1] \cong N \times [0, 1]$, in particular the boundary components are diffeomorphic, hence $M \cong N$.

Corollary. *There is a unique smooth structure on D^n for $n > 5$.*

Proof. If M is a contractible compact manifold with a simply connected boundary of dim $n > 5$ then we can take the submanifold of 4.1 to be a point $* \to M$ and then applying the theorem we get that $M \cong * \times D^n \cong D^n$, the standard disk.

Given a diffeomorphism $f : S^n \to S^n$ then we define the twisted sphere

$$\Sigma_f \ := \ D^{n+1} \cup_f D^{n+1}$$

as the pushout. It can be shown via Morse theory that this is homeomorphic to S^{n+1} [Mil, Prop B, pg. 110], and it is a fact that all smooth structures of spheres are given in this way:

Lemma. *Every homotopy $n > 5$ sphere Σ is diffeomorphic to a twisted sphere.*

Proof. Consider a homotopy sphere Σ, we can always decompose a manifold by cutting out a submanifold, and gluing it back in via the inclusion of the boundry, so here we simply cut out a disk

$$\Sigma = (\Sigma - D^n) \cup_i D^n$$

Now it is clear that a homotopy sphere minus a disk is *homeomorphic* to a disk, however by the unique smooth structure on a disk we know that $\Sigma - D^n$ is *diffeomorphic* to the disk. Thus we have decomposed Σ into the pushout of two discs, along a diffeomorphism.

Corollary (Generalized Poincare Conjecture)**.** *A homotopy $n > 5$ sphere is homeomorphic to the standard sphere.*

Proof. This is immediate from the Lemma and [Mil, Prop B, pg. 110], as every homotopy sphere is diffeomorphic to a twisted sphere and twisted spheres are all homeomorphic to the standard sphere.

From now on all manifolds are smooth and oriented. All maps are orientation preserving. Smooth structures means smooth structures *with orientation* up to orientation preserving diffeomorphism and cobordism is oriented cobordism. [Kos93] helps to clarify when orientation is assumed or not. If we denote Θ_n the set of h-cobordism classes of homotopy n-spheres then we can combine these facts to see that up to homeomorphism this set contains only *actual topological* spheres. By h-cobordism Θ_n is in bijection with the diffeomorphism classes of homotopy spheres, that is diffeomorphism classes of S^n. So Θ_n is just the set of differentiable structures on S^n.

$$\Theta_n \cong \text{DiffStruc}(S^n)$$

as sets. In fact they are isomorphic groups, this is in [Kos93, Thm 5.1].

Remark. Because all the spaces in the h-cobordism theorem are assumed to be simply connected they are also orientable. Moreover because they are subspaces of one another we can assume that the orientations are compatible and that the maps are orientation preserving diffeomorphisms.

2.1 Lower Dimensions

I beleive all of these results can be extended to $n = 5$. In dimension 4 the results are much more pathalogical, there are infinite smooth structures on $\mathbb{R}^4$. The Poincare conjecture is now known for all n. The case of $n = 4$ was done by Freedman in 1982 and earned him a Fields medal, the $n = 3$ case was done by Perlman in 2003 worthy of another Fields medal. There is a unique smooth structures on disks and spheres in dimensions $1 - 3$, this follows from classifications of surfaces and Perlmans results plus a result of Moise that there is a unique smooth structure on a closed 3 manifold. Thus for all dimensions other than 4 we have the bijection on cobordism groups and smooth structures on spheres. In dimension 4 the smooth Poincare conjecture is still open, that is we dont know how many smooth structures there are on S^4 (one, more than one or infinite). It is known that $\Theta_4 = 0$ so the bijection is unknown here, what is known is that the twisting spheres construction does not work, namely it is known that if we glue two 4 discs along a diffeomorphism we always get something diffeomorphic to the standard sphere. So in particular we have that

$$0 = \Theta_4 \cong \Gamma_4 \overset{?}{\leftrightarrow} \mathrm{DiffStruc}(S^4)$$

where Γ is just the image of the twisted spheres construction.

3 Groups of Diffeomorphisms

All diffeomorphisms are orientation preserving. So far we have seen two ways to view the group DiffStruc(S^n), as the cobordism group of homotopy spheres and as given by diffeomorphisms of S^{n-1}, gluing discs together along them. We have seen that the map

$$\mathrm{Diff}(S^{n-1}) \to \mathrm{DiffStruc}(S^n)$$

$$f \mapsto D^n \cup_f D^n$$

is surjective. We claim that its kernel is maps psuedo-isotopic to the identity, that is that we have a bijection

$$\tilde{\pi}_0 \mathrm{Diff}(S^{n-1}) \xrightarrow{\sim} \mathrm{DiffStruc}(S^n)$$

Proof. By [Kup, Lem 23.2.2] this map is a group homomorphism. We already know that it is surjective so it suffices to prove injectivity, and hence suffices to prove that the kernel is trivial, i.e. it reduces to proving that the kernel is maps psuedo-isotopic to the identity, which is equivalent to saying that $S_f := D^n \cup_f D^n$ is diffeomorphic to the standard sphere iff f is pseudo isotopic to the identity (proven here [Kup, Prop. 23.2.3]).

$\Longleftarrow$: Assume f is psudeo isotopic to the identity via the map F. If we consider S_f as two disks glued along their boundary then we can stretch the center out into a cylinder (image) and apply which is then topologically $S^{n-1} \times I$. We can therefore apply F to this section which maps it diffeomorphically to itself, however its image will be given by $F(-, 1) = id$ and hence the smooth structure will be the standard smooth structure. We can extend F off the cylinder by the identity (simply by definition on one end of the cylinder it extends because the smooth structure there is given by f and therefore agrees with F, and the same on the other side) thus producing a diffeomorphism of the total space to the standard sphere.

$\Longrightarrow$: Begin with a diffeomorphism $S_f \to S^n$, WLOG that is orientation preserving. Again we take the center of S_f and stretch it into a cylinder. Then the two ends of the cylinder are given by the glued disks, and their inclusion composed with the diffeomorphism to S^n is isotopic to the standard inclision into S^n, and moreover it may be arranged such that it commutes with the given diffeomorphism. This inclusion is isotopic to the identity because the space of inclusions of two points into S^n is path connected (//aparently). Thus the diffeomorphism may be interpreted as a diffeomorphism of the central cylinder and therefore as an isotopy between the identity and f. A little vague.

This group can be related to the diffeomorphisms of a disc relative to the boundary, i.e. diffeomorphisms of D^n that fix pointwise a neighborhood of the boundary, we denote this group $\mathrm{Diff}_\partial(D^n)$, and claim that

$$\tilde{\pi}_0 \mathrm{Diff}_\partial(D^n) \to \tilde{\pi}_0 \mathrm{Diff}(S^n)$$

given by extending the map by the identity, we think of the sphere as two disks glued and because the diffeomorphisms here fix the boundary pointwise, we may extend over the other half of the sphere.

Proof. Reference? I beleive that the key is that the following is a fiber sequence of topological spaces

$$\mathrm{Diff}_\partial(D^n) \to \mathrm{Diff}(S^n) \to \mathrm{Emb}(D^n, S^n)$$

The last map is removing a disc (half the sphere), that is if we have a diffeo $S^n \to S^n$ then we get an embedding $S^n -$ northern hemisphere $\cong D^n \to S^n$. The fiber over the identity of this map is given by all the diffeomorphisms of S^n such that removing a disc is the identity, however this is just exactly the diffeomorphisms of a disk that are identity on (a neighbourhood of) the boundardy, as we remove the interior and this gives us the identity on the boundary that we extend over the *other* half of the sphere.

Now we can (loosely, because I have not proven a LES for *psuedo-isotopy classes*, although it should be similar) apply the LES for homotopy groups to get that

$$\cdots \to \tilde{\pi}_1\mathrm{Emb}(D^n, S^n) \to \tilde{\pi}_0\mathrm{Diff}_\partial(D^n) \to \tilde{\pi}_0\mathrm{Diff}(S^n) \to \tilde{\pi}_0\mathrm{Emb}(D^n, S^n)$$

Finally the result would follow from the fact that $\mathrm{Emb}(D^n, S^n)$ is contractible (and thus its homotopy groups are all zero).

The fact that $\tilde{\pi}_0\mathrm{Emb}(D^n, S^n) = 0$ was also used in the previous lemma without proof and I know nothing about the $\tilde{\pi}_1$ group.

In the case of diffeomorphisms of discs there is no distinction between psuedo-isotopy and isotopy, in particular the map

$$\pi_0\mathrm{Diff}_\partial(D^n) \to \tilde{\pi}_0\mathrm{Diff}_\partial(D^n)$$

given by identifying the maps up to psudeo-isotopy is a bijection

Proof. [Kup, Cor. 23.1.6]. The proof uses a highly non-trivial fact that the so called "concordance diffeomorphism" [Kup, Def. 21.3.1] group is contractable for a disk [Kup, Lem 23.1.2].

3.1 Summary

Thus in high dimensions we have that

$$\pi_0\mathrm{Diff}_\partial(D^{n-1}) \cong \tilde{\pi}_0\mathrm{Diff}_\partial(D^{n-1}) \cong \tilde{\pi}_0\mathrm{Diff}(S^{n-1}) \cong \mathrm{DiffStruct}(S^n) \cong \Theta_n$$

In fact this holds for all $n \neq 4$, however the lower dimensions need to be taken care of seperately.

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